$f(\sqrt{S_0})$

x	$(0, \sqrt{S_0})$	$\sqrt{S_0}$	$(\sqrt{S_0}, +\infty)$
$f'(x)$	$-$	0	$+$
f	$\downarrow$	absolute min	$\uparrow$

Thus the minimal perimeter occurs when $x = \sqrt{S_0}$, i.e. it is a square. ■

minimal perimeter $= 2\left(\sqrt{S_0} + \frac{S_0}{\sqrt{S_0}}\right)$

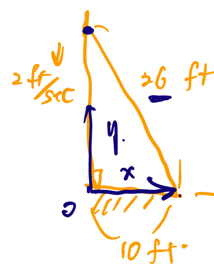
$= 4\sqrt{S_0}.$

8.2 Related Rates

Given rate of change of one quantity A, find the rate of change of another quantity B which is related to A. This is an application of implicit differentiation.

Example 8.2.1. A 26-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?

$\frac{dy}{dt}$ $\frac{dx}{dt}$ when $x = 10$ (unit ft)



Solution. At any time t, let

$x(t)$ = the distance of the bottom of the ladder from the wall
$y(t)$ = the distance of the top of the ladder from the ground

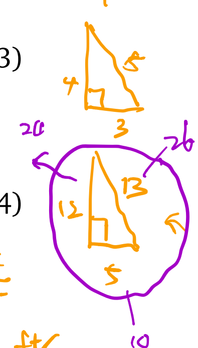
x and y are related by the Pythagorean relationship: solve for y

y:

$x = 10$

$$x^2(t) + y^2(t) = 26^2 \tag{8.3}$$

Recall:



Differentiating the above equation implicitly with respect to t, we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0. \tag{8.4}$$

10 24 −2

$2 \cdot 10 \cdot \frac{dx}{dt} - 2 \cdot 24 \cdot 2 = 0$

$\frac{dx}{dt} = +\frac{24}{5}$ ft/sec

The rates $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are related by equation (8.4). This is a related-rates problem.

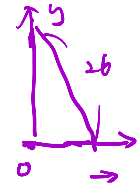
By assumption, $\frac{dy}{dt} = -2$ (y is decreasing at a constant rate of 2 feet per second).

When $x(t) = 10$, $y(t) = \sqrt{26^2 - 10^2} = 24$ feet.

So,

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = \frac{-2(24)(-2)}{2(10)} = 4.8 \text{ feet per second.}$$

The bottom of the ladder is moving away from the wall at a rate of 4.8 feet per second. ■



$\frac{dx}{dt} = 3$ ft/sec.

Exercise 8.2.1. Again, a 26-foot ladder is placed against a wall. If the bottom of the ladder is moving away from the wall at 3 feet per second, at what rate is the top moving down when the top of the ladder is 24 feet above ground?

$y = 24$

Find: $-\frac{dy}{dt}$

$x^2 + y^2 = 26^2$

$y = 24$

$x = 10$

Procedure to Solve Related Rates Problems

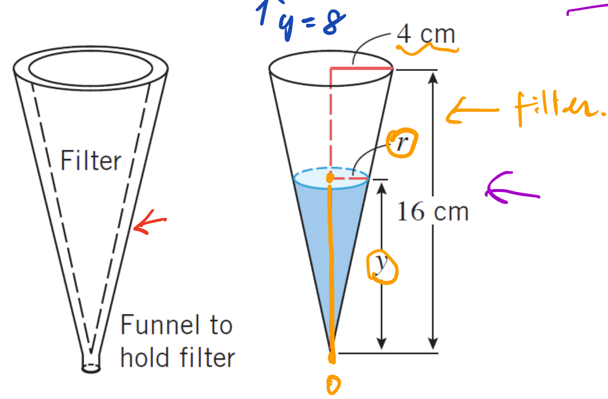
1. Assign variables: independent (usually t), dependent (usually $x, y, \ldots$)
2. Find the relation between variables $x, y, \ldots$
3. Take implicit differentiation about t, solve the unkonw rate of change using the known rate of change.

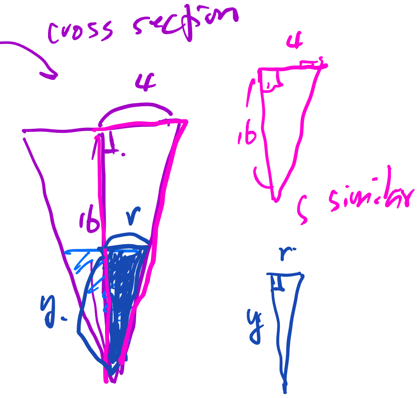
$\frac{dV}{dt} = -2$ cm^3/min.

Example 8.2.2. Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top. Suppose also that the liquid is forced out of the cone at a constant rate of 2 cm^3/min. What is the rate of change of the depth of liquid when the liquid is 8 cm deep?

Find $\frac{dy}{dt}$:

$y = 8$





$\frac{4}{16} = \frac{r}{y}$

$\frac{1}{4} y = r$

Solution. Let

t	=	time elapsed from the initial observation (min)
V	=	volume of liquid in the cone at time t (cm^3)
y	=	depth of the liquid in the cone at time t (cm)
r	=	radius of the liquid surface at time t (cm)

From the formula for the volume of a cone, the volume V, the radius r, and the depth y are related by

$$V = \frac{1}{3}\pi r^2 y = \frac{1}{3}\pi \frac{y^3}{16} \tag{8.5}$$

We are studying the related rate of V and y, so we eliminate r using similar triangle properties.

$$\frac{r}{y} = \frac{4}{16} \qquad \text{or } r = \frac{1}{4}y$$

Substituting this expression in (8.5) gives

$$V = \frac{\pi}{48}y^3$$

Differentiating both sides with respect to t we obtain $\frac{d}{dt}$

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2\frac{dy}{dt}\right)$$

$\frac{dV}{dt} = -2$ cm³/min at t of interest

$= \frac{\pi}{48}\left(3 \cdot 8^2 \frac{dy}{dt}\right)$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2} \tag{8.6}$$

$\frac{dy}{dt} = -2 \cdot \frac{48}{\pi} \frac{1}{3 \cdot 8^2}$

which expresses $\dfrac{dy}{dt}$ in terms of y.

The rate at which the depth is changing when the depth is 8 cm can be obtained from (8.6) with $y = 8$:

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min}$$

So, the depth is decreasing with rate $-\frac{1}{2\pi}$ cm/min at that moment.

■

MATH1520 University Mathematics for Applications	Spring 2021

Chapter 9: Indefinite Integrals

Learning Objectives:
(1) Compute indefinite integrals.
(2) Use the method of substitution to find indefinite integrals.
(3) Use integration by parts to find integrals and solve applied problems.
(4) Explore the antiderivatives of rational functions.

9.1 Antiderivatives

Definition 9.1.1. A function $F(x)$ is called an antiderivative of $f(x)$ if

$$F'(x) = f(x)$$

for every x in the domain of $f(x)$.

$(x^3)' = 3x^2.$

Example 9.1.1.

$\left(\frac{x^3}{3}\right)'$ $(5x)'$ $F(x) = \frac{x^3}{3} + 5x$ $F'(x) = x^2+5$
is an antiderivative of $f(x)$

1. $F(x) = \frac{1}{3}x^3+5x+2$ is an antiderivative of $f(x) = x^2+5$, since $F'(x) = (\frac{1}{3}x^3+5x+2)' = x^2 + 5$.
2. e^x is an antiderivative of e^x, since $(e^x)' = e^x$.

$f = e^x$ $F(x) = e^x \rightsquigarrow F' = f.$

A general antiderivative has the form $e^x + C$.
i.e. $F' = f$

Theorem 9.1.1 (Fundamental Property of Antiderivatives). *If $F(x)$ is an antiderivative of $f(x)$, then all antiderivative of $f(x)$ can be written as*

$$F(x) + C, \quad C \text{ is an arbitrary constant.}$$

$(F+C)' = f$

Proof. 1. For any constant C,

$$(F(x) + C)' = F'(x) = f(x),$$

so, $F(x) + C$ is an antiderivative of $f(x)$.

F, G are both antiderivatives of f(x) i.e. $F' = f = G' \rightsquigarrow F' - G' = 0$

$\quad\quad = (F-G)' = 0$

$F-G$ is constant function $= C$

2. For any antiderivative $G(x)$ with $G'(x) = f(x)$,

$$(G(x) - F(x))' = f(x) - f(x) = 0,$$

then, $G(x) - F(x) = C$ for some constant C.

$\rightsquigarrow G = F + C$

Thus, the general antiderivative of $f(x)$ is $F(x) + C$, $C \in \mathbb{R}$. □

Definition 9.1.2. The indefinite integral of $f(x)$ is the collection of all antiderivatives of $f(x)$, denoted by

$$\int f(x)\, dx,$$

$\int$ $\quad d \rightsquigarrow \Delta$ $\quad \int \rightsquigarrow \Sigma$

where $\int$ is the integral symbol, $f(x)$ is the integrand, and dx identifies x as the variable of integration.

The process of finding all antiderivatives is called indefinite integration.

Remark. It is useful to remember that if you have performed an indefinite integration calculation that leads you to believe that $\int f(x)\, dx = G(x) + C$, then you can check your calculation by differentiating $G(x)$:

If $G'(x) = f(x)$, then the integration $\int f(x)\, dx = G(x) + C$ is correct, but if $G'(x)$ is anything other than $f(x)$, you've made a mistake.

$$\boxed{F'(x) = f(x) \quad \Rightarrow \quad \int f(x)\, dx = F(x) + C}$$

The fact that indefinite integration and differentiation are reverse operations, except for the addition of the constant of integration, can be expressed symbolically as

$$\frac{d}{dx}\left[\int f(x)\, dx\right] = f(x)$$

and

$$\int F'(x)\, dx = F(x) + C.$$

9.2 Basic integration formulas

The relationship between differentiation and antidifferentiation enables us to establish the following integration rules by "reversing" analogous differentiation rules.

Theorem 9.2.1.

Differential Calculus:

$(k)' = 0$ for any constant k.

$(x^m)' = m x^{m-1}$ $\left(\frac{x^m}{m}\right)' = x^{m-1}$

$\left(\frac{x^{n+1}}{n+1}\right)' = x^n$

1. $\int k\,dx = kx + C$ *for constant* k.
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ *for all* $n \neq -1$
3. $\int \frac{1}{x}\,dx = \ln|x| + C$ *for all* $x \neq 0$. $(\ln|x|)' = \frac{1}{x}$
4. $$\int e^x\,dx = e^x + C,$$ $$\int a^x\,dx = \frac{1}{\ln a}a^x + C \qquad a > 0, a \neq 1.$$

$(a^x)' = (\ln a)a^x$ $\left(\frac{a^x}{\ln a}\right)' = a^x$

Theorem 9.2.2.

1. $\int kf(x)\,dx = k\int f(x)\,dx,$ *(constant multiple rule)*
2. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx,$ *(sum/difference rule)*

Caution: Both sides of the equality involve constant C.

Example 9.2.1.

1.
$$\int 3x^7 dx = 3\int x^7 dx$$
$$= 3 \cdot \frac{x^8}{8} + C.$$

2.
$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= \frac{1}{1/2}x^{1/2} + C.$$
$$= 2\sqrt{x} + C$$

for an arbitrary constant C

3.

$$\int (2x^5 + 8x^3 - 3x^2 + 5)\,dx = 2\int x^5\,dx + 8\int x^3\,dx - 3\int x^2\,dx + \int 5\,dx \quad \text{(No need to add } C\text{)}$$

$$= 2\left(\frac{x^6}{6}\right) + 8\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 5x + C \quad \text{(Add one } C\text{)}$$

constant

arbitrary constant.

$$= \frac{1}{3}x^6 + 2x^4 - x^3 + 5x + C.$$

4.

$$\int \left(\frac{x^3 + 2x - 7}{x}\right) dx = \int \left(x^2 + 2 - \frac{7}{x}\right) dx$$

$$= \frac{1}{3}x^3 + 2x - 7\ln|x| + C.$$

C: arbitrary constant.

5.

$$\int (3e^t + \sqrt{t})\,dt = \int (3e^t + t^{1/2})\,dt \;= 3\int e^t\,dt + \int t^{1/2}\,dt.$$

$$= 3\left(e^t\right) + \frac{1}{3/2}t^{3/2} + C$$

$$= 3e^t + \frac{2}{3}t^{3/2} + C.$$

Exercise 9.2.1.

$$\int \frac{(x + \sqrt{x})(x+1)}{\sqrt{x}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} + x + C$$

y line

Example 9.2.2. Find the function $f(x)$ whose tangent has slope $4x^3 + 5$ for each value of x and whose graph passes through the point (1, 10).

Solution. The slope of the tangent at each point $(x, f(x))$ is the derivative $f'(x)$. Thus,

$$f'(x) = 4x^3 + 5$$

when $x=1$, $f(1) = 10$

and so $f(x)$ is the antiderivative

$$\int f'(x)\,dx = \int (4x^3 + 5)\,dx = x^4 + 5x + C.$$

plug in $x=1$

To find C, use the fact that the graph of f passes through $(1, 10)$. That is, substitute $x = 1$ and $f(1) = 10$ into the equation for $f(x)$ and solve for C to get

$$f(1) = 10 = (1)^4 + 5(1) + C \qquad \text{or} \qquad C = 4.$$

Thus, the desired function is $f(x) = x^4 + 5x + 4$. ■

9.3 Integration by Substitution

chain rule in differential calculus.

$f(x) = g(u(x))$

$\frac{df}{dx} = \frac{dg}{du}\frac{du}{dx}$

Motivation

Let $f(x) = (x^2+3x-5)^{10}$. We can compute $f'(x)$ using the Chain Rule. It is:

$$f'(x) = 10(x^2+3x-5)^9 \cdot (2x+3) = (20x+30)(x^2+3x-5)^9.$$

$g(u) = f + C = \int \frac{df}{dx}\,dx = \int \frac{dg}{du}\frac{du}{\cancel{dx}}\cancel{dx}$

$\int \frac{dg}{du}\,du = g(u) + C$

Conversely, we have

$$\int (20x+30)(x^2+3x-5)^9\,dx = (x^2+3x-5)^{10} + C.$$

How would we obtain this indefinite integral without starting with $f(x)$?

Let $u = x^2+3x-5$. Thus

$$\frac{du}{dx} = 2x+3, \quad \text{or} \quad du = (2x+3)dx.$$

$u = x^2+3x-5.$ $\quad \frac{du}{dx} = (2x+3)$

Therefore,

$$\begin{aligned}
\int (20x+30)(x^2+3x-5)^9\,dx &= \int 10(2x+3)(x^2+3x-5)^9\,dx \\
&= \int 10\underbrace{(x^2+3x-5)}_{u}{}^9\,\underbrace{(2x+3)\,dx}_{du} \\
&= \int 10u^9\,du \\
&= u^{10} + C \quad \text{(replace } u \text{ with } x^2+3x-5) \\
&= (x^2+3x-5)^{10} + C
\end{aligned}$$

$g'(u) = 10u^9$

$g(u) = u^{10} + C$

$= \int 10 \cdot \frac{du}{dx} \cdot u^9\,dx$

$= \int 10u^9 \frac{du}{dx}\,dx$

$= g(u) = u^{10} + C$

$= (x^2+3x-5)^{10} + C$ □

More generally, we have

Theorem 9.3.1 (Integration by Substitution)**.**

$$\boxed{\int f(g(x))g'(x)\,dx \overset{u=g(x)}{=\!=} \int f(u)\,du}$$

Key idea: Make a guess $u = g(x)$, realize the integrand as a product of $f(u)$ and $u'(x)$.

Example 9.3.1.

$$\int (2x+1)^{2019}dx.$$

$u = 2x+1$

$\frac{du}{dx} = 2.$

formally $du = 2dx$

$\frac{1}{2}du = dx$

$\int u^{2019}\,dx$

$= \int u^{2019} \frac{du}{dx}/2\,dx = \frac{1}{2}\int u^{2019}\,du.$

$= \frac{1}{2} \frac{u^{2020}}{2020} + C$

$= \frac{(2x+1)^{2020}}{4040} + C.$

Solution. Let $u = g(x) = 2x + 1$, $f(u) = u^{2019}$. Then $du = 2dx$.

$$\begin{aligned}
\int (2x+1)^{2019} dx &= \frac{1}{2} \int \underbrace{(2x+1)^{2019}}_{f(g(x))} \cdot \underbrace{2}_{g'(x)} dx \\
&= \frac{1}{2} \int u^{2019}\, du \\
&= \frac{u^{2020}}{2 \times 2020} + C \\
&= \frac{(2x+1)^{2020}}{4040} + C.
\end{aligned}$$

Remark: usually, it is more convenient to write:

$$\begin{aligned}
\int (2x+1)^{2019} dx &= \int u^{2019} \frac{1}{2} du \qquad \left(\frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2} du\right) \\
&= \frac{u^{2019}}{2 \times 2020} + C \\
&= \frac{(2x+1)^{2020}}{4040} + C.
\end{aligned}$$

■

Example 9.3.2. Evaluate $\int \frac{7}{-3x+1} dx$. $= 7 \int \frac{1}{-3x+1} dx$

u

$u := -3x+1$

$du = -3dx$

$-\frac{1}{3} du = dx$

Solution. Let $u = -3x + 1$, then $\frac{du}{dx} = -3$, $dx = -\frac{1}{3} du$.

$$\begin{aligned}
\int \frac{7}{-3x+1} dx &= \int \frac{7}{u} \frac{du}{-3} \\
&= \frac{-7}{3} \int \frac{du}{u} \\
&= \frac{-7}{3} \ln|u| + C \\
&= -\frac{7}{3} \ln|-3x+1| + C.
\end{aligned}$$

■

$7 \int \frac{1}{u} \left(-\frac{1}{3} du\right)$

$= \frac{-7}{3} \int \frac{du}{u} = \frac{-7}{3} \ln|u| + C$

$= \frac{-7}{3} \ln|-3x+1| + C$

□.

Example 9.3.3. Evaluate $\int x\sqrt{x+3}\,dx$.

Solution. Let $u = x+3$, then $x = u-3$, $dx = du$, so,

$$\begin{aligned}\int x\sqrt{x+3}\,dx &= \int (u-3)u^{\frac{1}{2}}\,du \\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)du \\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

■

Exercise 9.3.1.

let $u = 3x+1 \rightarrow du = 3\,dx$ $\quad dx = \frac{1}{3}du$

1. $\int \sqrt{3x+1}\,dx = \frac{2}{9}(3x+1)^{\frac{3}{2}} + C$

u.

$= \int \sqrt{u}\cdot\frac{1}{3}du = \frac{1}{3}\int u^{\frac{1}{2}}du = \frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \frac{2}{9}(3x+1)^{3/2} + C$ □

2. $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$, where $a \neq 0$.

3. $\int x(x-1)^{100}\,dx = \frac{1}{102}(x-1)^{102} + \frac{1}{101}(x-1)^{101} + C$

let $u = x-1$

$x = u+1$

$du = dx$

$\int (u+1)u^{100}du = \int (u^{101} + u^{100})du = \int u^{101}du + \int u^{100}du$

$= \frac{u^{102}}{102} + \frac{u^{101}}{101} + C$

$= \frac{(x-1)^{102}}{102} + \frac{(x-1)^{101}}{101} + C$ □

Example 9.3.4. Evaluate $\int xe^{x^2+5}\,dx$

Solution. Let $u = g(x) = x^2+5$, hence $du = 2x\,dx$

$$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$\int e^u x\,dx = \int \frac{e^u du}{2} = \frac{1}{2}e^u + C$

$$\begin{aligned}\int xe^{x^2+5}\,dx &= \int e^{\overbrace{x^2+5}^{u}}\underbrace{x\,dx}_{\frac{1}{2}du} \\ &= \int \frac{1}{2}e^u\,du\end{aligned}$$

$= \frac{1}{2}e^{x^2+5} + C$ □

$$= \frac{1}{2}e^{u} + C \quad \text{(now replace } u \text{ with } x^2+5\text{)}$$
$$= \frac{1}{2}e^{x^2+5} + C.$$

Remark: Sometimes, we even do not need to introduce the new variable u, just keep in mind which part should be regarded as $u = g(x)$.

$$\int xe^{x^2+5}\,dx = \int \frac{1}{2}e^{x^2+5}\,d(x^2+5) \quad (\text{Regard } u = x^2+5)$$
$$= \frac{1}{2}e^{x^2+5} + C.$$

■

Example 9.3.5. Evaluate $\int x^3\sqrt{x^4+1}\,dx$

let $u = x^4+1$
$du = 4x^3\,dx$
$\frac{du}{4} = x^3\,dx$

Solution.

$\sqrt{u}$.

$$\int x^3\sqrt{x^4+1}\,dx = \int \frac{1}{4}\sqrt{x^4+1}\,d(x^4+1) \quad (\text{Regard } u = x^4+1)$$
$$= \frac{1}{6}(x^4+1)^{3/2} + C.$$

$\int \sqrt{u}\,\frac{du}{4} =$

$\frac{du}{4}$

■

$= \frac{1}{4}\int u^{\frac{1}{2}}\,du = \frac{1}{4}\cdot u^{3/2}/\tfrac{3}{2} + C = \frac{1}{6}(x^4+1)^{\frac{3}{2}} + C$

□.

Example 9.3.6. Evaluate $\int \frac{1}{x\ln x}\,dx$

let $u = \ln x$. ⇝ $du = \frac{1}{x}\,dx$

Solution.

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\,d(\ln x) (\text{Regard } u = \ln x)$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|u| + C$$
$$= \ln|\ln x| + C.$$

$\int \frac{1}{u}\,du$
$= \ln|u| + C$
$= \ln|\ln x| + C$

Remark: To avoid mistakes, we can take the derivative to verify our answer. ■

Exercise 9.3.2.

1. $\displaystyle\int x^3 e^{x^4}\,dx = \frac{1}{4}e^{x^4} + C.$
2. $\displaystyle\int 6x\sqrt{x^2+3}\,dx = 2(x^2+3)^{\frac{3}{2}} + C.$
3. $\displaystyle\int e^x\sqrt{e^x+1}\,dx = \frac{2}{3}(e^x+1)^{\frac{3}{2}} + C.$

   let $u = e^x + 1$ $\quad du = e^x\,dx$

   du

   $\displaystyle\int \sqrt{u}\,du = \frac{u^{\frac{3}{2}}}{3/2} + C = \frac{2}{3}(e^x+1)^{3/2} + C$ $\square$
4. $\displaystyle\int (2x-1)(x^2-x)^{100}\,dx = \frac{1}{101}(x^2-x)^{101} + C$

9.4 Integration by Parts

Motivation

Let $u(x)$ and $v(x)$ be differentiable functions. By the product rule, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides with respect to x,

$$\begin{aligned}\int u\frac{dv}{dx}\,dx &= \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx\\ &= uv - \int v\frac{du}{dx}\,dx\end{aligned}$$

which is

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

or

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Key Idea: Write the integrand as product of $u(x)$ and $v'(x)$, then integrate by parts.

Example 9.4.1. Compute $\displaystyle\int xe^x\,dx$.